# BAYESIAN ESTIMATION OF THE RELIABILITY FUNCTION USING THE POWER RULE MODEL

BY

Abdel Hamid M.Nigm
Statistics Dept.
Faculty of Science
Jidda

Moshira A. Ismail
Statistics Dept.
Faculty of Economics
Cairo University

Mohamed A. Al Kobaty
Statistics Dept.
Sana University

## ABSTRACT

Bayesian estimates of $R(t_0)$, the reliability function at a mission time $t_0$, at use conditions stress $v_u$ are considered. The two-parameter Weibull distribution and the inverse power rule model as a time transformation function in accelerated life testing are assumed. The estimates are obtained using vague prior information about the parameters. Numerical results are presented.

Key Words: Accelerated life testing, power rule model, Bayesian inference, reliability estimation, Weibull distribution .

## 1- INTRODUCTION

In practice long-life items are subjected to larger than usual stresses in order to obtain failure data in a short amount of test time. Such tests are called accelerated or overstress life tests, and a goal is to infer the life distribution of the items at the usual stress level using failure data from accelerated tests. In accelerated life testing experiments, $k$ random samples of size $n_i$, $i=1, 2, \dots k$ items are subjected to high stress level $v_i$, such that $0<v_u<v_1<\dots<v_K$ where $v_i<v_j$ denotes that $v_i$ is less severe than $v_j$ and $v_u$ is the stress at normal use. The resulting times to failure data are $t_{i1}<t_{i2}<\dots<t_{ir_i}$, $r_i \le n_i$. The case $r_i<n_i$, denotes a type II censored sample test, and $r_i=n_i$ denotes the special case of complete sampling We make

the following assumptions which are widely used in accelerated life testing:

1) The time to failure $T$ has the two-parameter Weibull probability density function (pdf)

$$f(t;\theta,\beta)=\left(\frac{\beta}{\theta}\right)\left(\frac{t}{\theta}\right)^{\beta-1}e^{-\left(\frac{t}{\theta}\right)^{\beta}} \quad , \qquad t,\theta,\beta>0 \tag{1.1}$$

The case $\beta=1$, leads to the one-parameter exponential distribution.

2) The scale parameter $\theta_i$, $i=1, 2, ..., k, u$ is inversely proportional with the $p^{th}$ power of stress $v_i$. That is by the power rule model

$$\theta_i = cx_i^{-p}\ ,\ c,x_i>0 \tag{1.2}$$

where $c$ is a constant of proportionality, $x_i=\left(v_i/\overset{*}{v}\right)$ and $\overset{*}{v}$ is the weighted geometric mean of the $v_i$'s. That is,

$$\overset{*}{v}=\prod_{i=1}^{k}\left[\left(v_i\right)^{r_i/\sum_{i=1}^{k}r_i}\right]$$

[see Singpurwalla (1971)]

3) The shape parameter $\beta$ is invariant with the stress level, that is it is independent of the stress level.

4) To ensure that there is no correlation between the tests, the selected stresses $v_i$'s are randomly arranged and tests are then conducted in accordance with this random sequence.

5) The time to failure at use conditions stress $v_u$ has the same parametric family of the time to failure at high stress level $v_i$.

Using a classical approach, Nigm, Ismail and Al-Kobaty (1991) considered point and interval estimates for the reliability function at a mission time $t_0$ at use conditions stress $v_u$, where $R(t_0)$ is given by

$$R(t_0)=e^{\left[-\left(\frac{t_0 x_u^p}{c}\right)^{\beta}\right]} \tag{1.3}$$

There is often prior information about $c$, $p$ and $\beta$. The main objective of this paper is to obtain estimates for $R(t_0)$ using a Bayesian approach which incorporates prior information about the parameters by treating them as random variables. In our analysis, the variability of $R(t_0)$

will depend on the variability of $c$, $p$ and $\beta$. Proschan & Singpurwalla (1980) considered Bayesian estimation for $R(t_0)$. They used a non-parametric approach which requires neither distributional assumptions nor the specification of a time transformation function. Also their Bayesian estimate of $R(t_0)$ was based on regression analysis. Willing (1987) used the pdf (1.1) with $\lambda_i=1/\theta_i$, $r_i=n_i$ and a specific power rule model $\lambda_i = \lambda_u e^{(pV_i^3)}$. He used an improper joint prior distribution for $\lambda_u$, $p$ and $\beta$ as $\prod(\lambda_u, p, \beta) \propto (\lambda_u p\beta)^{-1}$ to derive the corresponding posterior distribution of $\lambda_u$, $p$ and $\beta$ given the data. Extension can be made by a transformation technique to find Bayesian estimates for $R(t_0)$. Blackwell & Singpurwalla (1988) estimated $R(t)$ using the exponential distribution and the power rule model with inference using a Kalman filter with correlated observation errors. However extensions to other distributions involve non-linear filtering and were not considered.

We will consider the following estimators for $R(t_0)$

(i) The posterior mean

$$\overset{*}{R}(t_0) = E\left[R(t_0)\,|\,data\right] \tag{1.4}$$

where the expectation is with respect to the posterior distribution. Note that under the squared-error loss function $l\left[R(t_0), \tilde{R}(t_0)\right] = \left[R(t_0) - \tilde{R}(t_0)\right]^2$ where $\tilde{R}(t_0)$ is an estimate for $R(t_0)$. the posterior mean $\overset{*}{R}(t_0)$ is the choice of $\tilde{R}(t_0)$ which minimizes the posterior risk.

(ii) The Bayes estimator corresponding to the loss function

$$l\left[R(t_0), \tilde{R}(t_0)\right] = \left(\frac{\tilde{R}(t_0) - R(t_0)}{\tilde{R}(t_0)}\right)^2$$

In this case the Bayes decision $\dot{R}(t_0)$, the value of $\tilde{R}(t_0)$ which minimizes the posterior risk is given by

$$\dot{R}(t_0) = \frac{E\left[R^2(t_0)\,|\,data\right]}{E\left[R(t_0)\,|\,data\right]} \tag{1.5}$$

It is assumed that

$$E\left[R^2(t_0)|data\right] < \infty \quad \text{and} \quad E\left[R(t_0)|\, data\right] \neq 0$$

This loss function has been considered by other authors; for example De Groot (1970) and Ismail (1988) in different applications .

In Section 2 we will consider estimates of $R(t_0)$ for the Weibull distribution with unknown parameters. Section 3 deals with the case of known shape parameter. In Section 4, the results are illustrated numerically.

## 2- WEIBULL DISTRIBUTION WHERE $c, p$ AND $\beta$ ARE RANDOM VARIABLES

### *2.1- The Posterior Distribution.*

Let the set of data of the $k$ samples test be denoted by $\left\{\dot{t}_{ij}, i = 1,2,\ldots,k \;,\; j = 1,2,\ldots,r_i \leq n_i\right\}$, where for fixed $i$, $t_{i1} < t_{i2} < \ldots < t_{ir_i}$ and noting that $\sum_i r_i \log x_i = 0$. Then according to the assumptions of Section 1 the likelihood function of $\dot{t}_{ij}$ given $c, p, \beta$ can be written as

$$L\left(\dot{t}_{ij}|c,p,\beta\right) = \left(\prod_i a_i\right)\beta^{\sum_i r_i}\left(\frac{1}{c}\right)^{\beta\sum_i r_i}\left(\prod_i\prod_j t_{ij}\right)^{\beta-1} e^{-\left(\frac{1}{c}\right)^{\beta} A_1(p,\beta)}$$

where

$i = 1,2,\ldots,k \quad , \quad j = 1,2,\ldots,r_i \leq n_i$

$a_i = \dfrac{n_i!}{(n_i - r_i)!}$ and

$$A_1(p,\beta) = \sum_i\sum_j\left(t_{ij}x_i^p\right)^{\beta} + \sum_i (n_i - r_i)\left(t_{ir_i}x_i^p\right)^{\beta}$$

Consider an improper prior distribution for $c$ of the form $\pi(c) \propto 1/c \;,\quad c > 0$ and uniform prior distributions for $p$ and $\beta$ as

$$\pi(p) = \frac{1}{\gamma_1 - \gamma_0} \quad , \quad -\infty < \gamma_0 < p < \gamma_1 < \infty$$

$$\pi(\beta) = \frac{1}{\gamma_3 - \gamma_2} \quad , \quad -\infty < \gamma_2 < \beta < \gamma_3 < \infty$$

and let $c, p, \beta$ be apriori independent.

Then the posterior distribution of $c, p, \beta$ given the set of data $D = \left\{\dot{t}_{ij}, \gamma_0, \gamma_1, \gamma_2, \gamma_3\right\}$ is given by :

$$\dot{\pi}(c,p,\beta|D) = \left(\frac{1}{J_1 \Gamma\sum_i r_i}\right) \beta^{\sum_i r_i} \left(\prod_i \prod_j t_{ij}\right)^{\beta-1} \left(\frac{1}{c}\right)^{\left(\beta \sum_i r_i\right)+1} \times e^{\left\{-\left(\frac{1}{c}\right)^{\beta} A_1(p,\beta)\right\}} \qquad (2.1)$$

where $J_1 = \int_{\gamma_0}^{\gamma_1}\int_{\gamma_2}^{\gamma_3} \left\{ \beta^{\sum_i r_i - 1} \left(\prod_i \prod_j t_{ij}\right)^{\beta-1} \left[A_1(p,\beta)\right]^{-\sum_i r_i} \right\} d\beta\, dp$

and $\Gamma n$ is gamma $n$.

*2.2- Estimation based on the squared-error loss function:*

Using the posterior distribution (2.1), the posterior mean $\dot{R}(t_0)$ of $R(t_0) = exp\left[-\left(t_0 x_u^p / c\right)^{\beta}\right]$, where $x_u = (\nu_u / \dot{\nu})$ is given by

$$\dot{R}(t_0) = E\,[\,R(t_0) \mid D\,]$$

$$= \int_{\gamma_0}^{\gamma_1}\int_{\gamma_2}^{\gamma_3}\int_0^{\infty} exp\left[-\left(t_0 x_u^p / c\right)^{\beta}\right] \dot{\pi}(c,p,\beta|D)\, dc\, d\beta\, dp$$

$$= J_2(t_0) / J_1 \qquad (2.2)$$

In this case the minimum posterior risk (M.P.R) is the posterior variance given by

$$Var\left[R(t_0)|D\right] = E\left[R^2(t_0)|D\right] - \left\{E\left[R(t_0)|D\right]\right\}^2$$
$$= \left[J_3(t_0)/J_1\right] - \left[J_2(t_0)/J_1\right]^2 \qquad (2.3)$$

where, $J_1$ was defined in Subsection 2.1,

$$J_2(t_0)=\int_{\gamma_0}^{\gamma_1}\int_{\gamma_2}^{\gamma_3}\left(\beta^{\sum_i r_i-1}\left(\prod_i\prod_j t_{ij}\right)^{\beta-1}\left[A_1(p,\beta)+\left(t_0x_*^p\right)^{\beta}\right]^{-\sum_i r_i}\right)d\beta\, dp$$

and

$$J_3(t_0)=\int_{\gamma_0}^{\gamma_1}\int_{\gamma_2}^{\gamma_3}\left(\beta^{\sum_i r_i-1}\left(\prod_i\prod_j t_{ij}\right)^{\beta-1}\left[A_1(p,\beta)+2\left(t_0x_*^p\right)^{\beta}\right]^{-\sum_i r_i}\right)d\beta\, dp$$

### *2.3- Estimation based on the relative-error loss function:*

Using the posterior distribution (2.1), the Bayes decision $\dot{R}(t_0)$ of $R(t_0)$ is given by

$$\dot{R}(t_0)=E\left[R^2(t_0)|D\right]/E\left[R(t_0)|D\right]$$

In this case the minimum posterior risk is given by

$$\text{M.P.R}=\text{Var}\left[R(t_0)|D\right]/E\left[R^2(t_0)|D\right]$$

Then from the previous results

$$\dot{R}(t_0)=J_3(t_0)/J_2(t_0) \qquad (2.4)$$

and

$$\text{M.P.R}=[(J_3(t_0)/J_1)-(J_2(t_0)/J_1)^2]/(J_3(t_0)/J_1) \qquad (2.5)$$

## 3- WEIBULL DISTRIBUTION WITH KNOWN $\beta$

### *3.1- The posterior distribution:*

Consider the likelihood function of Subsection 2.1 where $\beta$ is a known fixed point, say $\beta_0$ and the same prior distributions of $c$ and $p$ defined in Subsection 2.1. Then the posterior distribution of $c$, $p$ given the set of data $D=\left\{\dot{t}_{ij}; i=1,2,3,\ldots,k\ ,\ j=1,2,\ldots,r_i\le n_i,\beta_0,\gamma_0,\gamma_1\right\}$ is given by

$$\dot{\pi}(c,p|D)=\frac{\beta_0}{J_4\Gamma\sum_i r_i}(1/c)^{\beta_0\sum_i r_i+1}\, e^{-(1/c)^{\beta_0}\sum_i A_1(\beta_0)}$$

where

$$A_i(\beta_0)=\sum_j t_{ij}^{\beta_0}+(n_i-r_i)t_{ir_i}^{\beta_0} \quad , \quad x_i=\left(v_i/\overset{\bullet}{v}\right)$$

and

$$J_4=\int_{\gamma_0}^{\gamma_1}\left\{\sum_i x_i^{\beta_0 p} A_i(\beta_0)\right\}^{-\sum_i r_i} dp$$

*3.2- Estimation based on the squared-error loss function:*

Using the posterior distribution (3.1), the posterior mean $\overset{\bullet}{R}(t_0)$ of $R(t_0)=exp\left[-\left(t_0 x_u^p/c\right)^{\beta_0}\right]$, where $x_u=(v_u\,/\,\overset{\bullet}{v})$ is given by

$$\overset{\bullet}{R}(t_0)=J_5(t_0)/J_1 \qquad (3.2)$$

with the posterior variance

$$\text{Var}\,[R(t_0)|D]=[J_6(t_0)/J_4]-[J_5(t_0)/J_4]^2 \qquad (3.3)$$

where, $J_4$ was defined in Subsection 3.1,

$$J_5(t_0)=\int_{\gamma_0}^{\gamma_1}\left\{\left(t_0 x_u^p\right)^{\beta_0}+\sum_i x_i^{\beta_0 p} A_i(\beta_0)\right\}^{-\sum_i r_i} dp$$

and

$$J_6(t_0)=\int_{\gamma_0}^{\gamma_1}\left\{2\left(t_0 x_u^p\right)^{\beta_0}+\sum_i x_i^{\beta_0 p} A_i(\beta_0)\right\}^{-\sum_i r_i} dp$$

*3.3- Estimation based on the relative-error loss function:*

Using the posterior distribution (3.1) the Bayes decision $\overset{\bullet}{R}(t_0)$ of $R(t_0)$ is given by

$$\overset{\bullet}{R}(t_0)=J_6(t_0)/J_5(t_0) \qquad (3.4)$$

and the minimum posterior risk is given by

$$\text{M.P.R.}=[(J_6(t_0)/J_4)-(J_5(t_0)/J_4)^2\,]/[J_6(t_0)/J_4\,] \qquad (3.5)$$

Note that the results of the exponential distribution can be obtained as a special case of the results in this Section if we set $\beta_0=1$ .

## 4- NUMERICAL RESULTS

Assume that the following data represent the high stress levels and the failure times of the items at each level of stress treated as samples from a two-parameter Weibull population where $n_i= 15$ and $r_i=12$ for $i=1,2,3,4$. and $v_u= 0.80$

| **Stress ($v_i$)** | **Failure times ($t_{ij}$)** |
|---|---|
| 0.87 | 1.67, 2.2, 2.51, 3.0, 3.9, 4.7, 4.7, 7.53, 14.7, 27.8, 37.4, 44.7, 52.0 |
| 0.99 | 0.8, 1.0, 1.37, 2.25, 2.95, 3.7, 6.07, 6.65, 7.05, 7.37, 8.254, 9.138 |
| 1.09 | 0.012, 0.18, 0.20, 0.24, 0.26, 0.32, 0.32, 0.42, 0.44, 0.88, 1.02, 1.25 |
| 1.18 | 0.073, 0.098, 0.117, 0.135, 0.175, 0.262, 0.270, 0.350, 0.386, 0.456, 0.456, 0.531 |

The results depend on the computation of the values of the integrals $J_1$ ,$J_2$ $(t_0)$ and $J_3$ $(t_0)$, defined in Section 2, by numerical integration techniques (Simpson-Rule was used). The ranges of the prior distribution of $p$ and $\beta$ are important for these computations where $\gamma_0 < p < \gamma_1$ and $\gamma_2 < \beta < \gamma_3$. In our case we have taken $0 < p < 20$ and $0 < \beta < 4$. The upper bound 20 in $p$ was chosen so as to cover a wide range. However, we observed that the value of the integrals depend on the range of $\beta$ rather than the range of $p$. Moreover, these integrals have negligibly small values when $\beta > 3.50$, which explains the choice of the upper bound of $\beta$ equal to 4. Also these integrals have very small values, so we multiplied the numerator and the denominator of each intergral by $(100)^{48}$.

To explain this, let us rewrite $J_1$ defined in Subsection 2.1 as

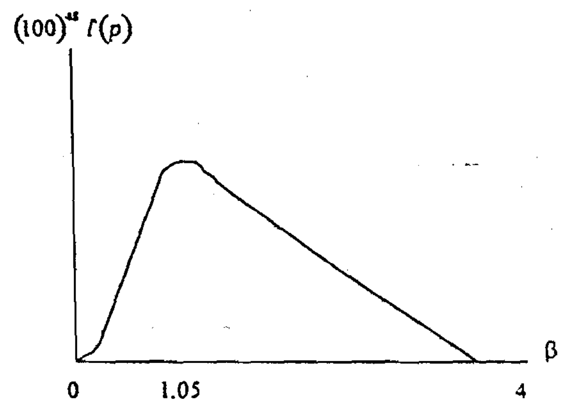
$$J_1 = \int_0^4 \int_0^{20} H(p,\beta)\,dp\,d\beta$$

Figure 4.1 shows the behavior of $\hat{I}(p)$, the approximate value of the inner integral

$$I(p) = \int_0^{20} H(p,\beta)\,dp$$

at specified values of $\beta$ calculated by Simpson-Rule. The ranges of $p$ and $\beta$ were divided into 80 parts.

**Fig 4.1: The behavior of $\hat{I}(p)$**




The estimates of $R(t_0)$ under squared-error loss function were computed using the formula (2.2) with the posterior variance defined by (2.3). The estimates of $R(t_0)$ under relative-error loss function were computed using the formula (2.4) with the minimum posterior risk defined by (2.5). Results of the two methods are shown in Table 4.1 .

**TABLE 4.1** : ***Squared-Error (S-E) and Relative-Error (R-E) Loss Functions Results.***

| $t_0$ | *Estimate of $R(t_0)$ (S-E)* | *Posterior Variance* | *Estimate of $R(t_0)$ (R-E)* | *Minimum Posterior Risk* |
|---|---|---|---|---|
| 5 | 0.96 | 0.00062 | 0.96 | 0.00068 |
| 10 | 0.91 | 0.00163 | 0.92 | 0.00194 |
| 15 | 0.88 | 0.00285 | 0.88 | 0.00361 |
| 20 | 0.84 | 0.00400 | 0.85 | 0.00562 |
| 25 | 0.81 | 0.00521 | 0.81 | 0.00794 |
| 30 | 0.77 | 0.00639 | 0.78 | 0.01056 |
| 35 | 0.74 | 0.00752 | 0.75 | 0.01346 |
| 40 | 0.71 | 0.00852 | 0.72 | 0.01663 |
| 45 | 0.68 | 0.00957 | 0.70 | 0.02005 |
| 50 | 0.66 | 0.01048 | 0.67 | 0.02374 |
| 55 | 0.63 | 0.01130 | 0.65 | 0.02766 |
| 60 | 0.61 | 0.01204 | 0.63 | 0.03183 |
| 65 | 0.58 | 0.01270 | 0.60 | 0.03623 |
| 70 | 0.56 | 0.01328 | 0.58 | 0.04086 |
| 75 | 0.54 | 0.01378 | 0.56 | 0.04571 |

Table 4.1 shows that as $t_0$ increases both the posterior variance and the minimum posterior risk increase which is expected since these measures reflect our degree of uncertainty about $R(t_0)$ .

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